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# Light-cone variables and the high energy limit of elastic scattering

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**Abstract.** A simple derivation is given of the eikonal approximation for high energy elastic scattering due to the multiple exchange of single scalar particles. This is generalized to the exchange of particles of arbitrary spin or multiple reggeons. The implications for very high energy are discussed.

## 1. Introduction

A theory of ultrahigh energy elastic scattering has been given by Cheng and Wu (1970) in a long series of papers in which summations are made of the leading terms in the high energy limit of various families of graphs in quantum electrodynamics. Other authors (Chang and Fishbane 1970, Chang and Yan 1970, Hasslacher *et al* 1970, Cheng and Wu 1971) have examined $\phi^3$ theories in addition to QED and have summed exchanges of basic primitive diagrams or 'towers'. The first calculation along these lines was performed in QED by Cheng and Wu (1969) and was concerned with electron–electron scattering via the exchange of any number of single photons, summed over all possible orderings of the end points. The result in all cases is the standard eikonal expression for scattering in an effective potential which is just the Fourier transform of a basic primitive diagram. For photon exchange (Cheng and Wu 1969) this is just the Coulomb potential. The authors remark that this simple result should be obtainable by a more direct argument. This has been partially supplied by Bjorken *et al* (1971) who discuss high energy scattering in an external field using light-cone quantization. In the following section we remark that for a free field the light-cone quantization of the annihilation and creation operators and the expansion of the field in terms of them differs only in the choice of variables from conventional quantization on a space-like surface. We then reproduce the Bjorken argument for the external field problem using the light-cone variables for the fields, but a conventional Feynman–Dyson interaction representation, based on time- (not $\tau$-) ordered products. This modification leads to a simple derivation of the eikonal formula for the elastic scattering of two hadrons interacting through the multiple exchange of spin zero mesons. This is presented in § 4, where it is also generalized to the exchange of particles of higher spin.

In a more complicated theory one may expect that the basic structure exchanged is something like a 'tower'. Since the exchanges of these 'towers' are in most cases exchanges of Regge poles (Lee and Sawyer 1962, Eden *et al* 1966), it is a simple matter to extend the eikonalization procedure to this case and so find the leading energy dependence. As an example (see Chang and Yan 1970, Hasslacher *et al* 1970), we do this for a purely absorptive Regge pole with a linear trajectory.

## 2. Free-field quantization

With covariant normalization the conventional commutation relations for the annihilation and creation operators of a free scalar field are

$$\delta(p^2-m^2)\theta(p_0)[a(p), a^\dagger(p')] = (2\pi)^3\delta^4(p-p'). \tag{2.1}$$

The corresponding expansion of the field operator is

$$\phi(x) = (2\pi)^{-3}\int d^4p\,\delta(p^2-m^2)\theta(p_0)(a(p)\,e^{-ipx}+b^\dagger(p)\,e^{ipx}), \tag{2.2}$$

where $b(p)$ and $b^\dagger(p)$ are the antiparticle operators, satisfying the same commutation relations as $a$ and $a^\dagger$. Expressions (2.1) and (2.2) can be evaluated on any space-like or light-like surface. For the latter we introduce the variables

$$2^{1/2}\tau = x^0+x^3, \qquad 2^{1/2}z = x^0-x^3 \tag{2.3}$$

and

$$2^{1/2}\eta = p^0+p^3, \qquad 2^{1/2}h = p^0-p^3. \tag{2.4}$$

Then

$$(p)^2-m^2 \equiv (p^0)^2-\epsilon^2(p) \equiv 2\eta h-\boldsymbol{p}^2-m^2, \tag{2.5}$$

where

$$\epsilon(p) = ((p^3)^2+\boldsymbol{p}^2+m^2)^{1/2} \tag{2.6}$$

and

$$\boldsymbol{p} = (p^1, p^2). \tag{2.7}$$

The commutator (2.1) may then be written in either of the alternative forms

$$\{a(\boldsymbol{p}, p^3), a^\dagger(\boldsymbol{p}', p^{3\prime})\} = (2\pi)^3 2\epsilon\delta^2(\boldsymbol{p}-\boldsymbol{p}')\,\delta(p^3-p^{3\prime}), \tag{2.8}$$

$$\{a(\boldsymbol{p}, \eta), a^\dagger(\boldsymbol{p}', \eta')\} = (2\pi)^3 2\eta\delta(\eta-\eta')\,\delta^2(\boldsymbol{p}-\boldsymbol{p}'), \qquad \eta > 0. \tag{2.9}$$

The former expression is the usual one for quantization on the surface $t = 0$, the latter for the light cone $\tau = 0$ (see Bjorken *et al* 1971).

Similarly the expansion of the field takes the alternative forms

$$\phi(x) = (2\pi)^{-3}\int\frac{d^3p}{2\epsilon}[a(\boldsymbol{p}, p^3)\exp\{-i(x^0\epsilon-p^3x^3-\boldsymbol{p}\,.\,\boldsymbol{x})\}$$
$$+b^\dagger(\boldsymbol{p}, p^3)\exp\{i(x^0\epsilon-p^3x^3-\boldsymbol{p}\,.\,\boldsymbol{x})\}] \tag{2.10}$$

or (Bjorken *et al* 1971)

$$\phi(x) = (2\pi)^{-3}\int d^2p\int_0^\infty\frac{d\eta}{2\eta}[a(\boldsymbol{p}, \eta)\exp\{-i(H\tau+\eta z-\boldsymbol{p}\,.\,\boldsymbol{x})\}$$
$$+b^\dagger(\boldsymbol{p}, \eta)\exp\{i(H\tau+\eta z-\boldsymbol{p}\,.\,\boldsymbol{x})\}] \tag{2.11}$$

where

$$H = \frac{p^2+m^2}{2\eta}. \tag{2.12}$$

In (2.9) and (2.11) left-moving particles (negative $p^3$) are contained in the limited range

$$0 \leqslant \eta \leqslant 2^{-1/2}m. \tag{2.13}$$

These expressions are thus most appropriate for high energy right-moving particles. The corresponding expressions appropriate to left-moving particles can be obtained integrating (2.1) and (2.2) over $\eta$ rather than $h$, which leads to

$$\{a(\boldsymbol{p}, h), a^{\dagger}(\boldsymbol{p}', h')\} = (2\pi)^3 2h\, \delta(h-h')\delta^2(\boldsymbol{p}-\boldsymbol{p}') \tag{2.14}$$

and

$$\phi(x) = (2\pi)^{-3}\int \mathrm{d}^2p \int_0^{\infty} \frac{\mathrm{d}h}{2h}\,[a(\boldsymbol{p}, h)\exp\{-\mathrm{i}(h\tau+\eta z-\boldsymbol{p}\,.\,\boldsymbol{x})\} \\ +b^{\dagger}(\boldsymbol{p}, h)\exp\{\mathrm{i}(h\tau+\eta z-\boldsymbol{p}\,.\,\boldsymbol{x})\}], \tag{2.15}$$

where

$$\eta = \frac{p^2+m^2}{2h}. \tag{2.16}$$

## 3. Scattering in an external field

Consider the scattering of a single scalar particle in a time-dependent classical external field $\phi^{\mathrm{ex}}(x)$. The interaction lagrangian density of the system is

$$L(x) = g\phi^{\dagger}(x)\phi(x)\phi^{\mathrm{ex}}(x). \tag{3.1}$$

If the field does not contain sufficiently high frequencies to create pairs, an elegant solution to the problem in the high energy limit has been given by Bjorken *et al* (1971). We present this in a slightly modified form which allows for a generalization below. We work in the conventional interaction representation based on space-like surfaces and the variable $x^0$, but express the fields in the right-moving form. The initial state can be taken to be that of a fast right-moving particle along the $x^3$ axis in the rest frame of the external field (ie that in which the significant momentum components of its Fourier transform are smallest in magnitude). The required $S$ matrix element is

$$S_{\mathrm{fi}} = \left\langle f \left| T\exp\left(-\mathrm{i}\int L(x)\,\mathrm{d}^4x\right)\right| \mathrm{i} \right\rangle, \tag{3.2}$$

where $T$ denotes the conventional time ordered product. Transform to the rest frame of the initial particle†

$$|\mathrm{i}\rangle = \exp(-\mathrm{i}\omega K_3)|\mathrm{i}_0\rangle, \tag{3.3}$$

† $K_3$ is the generator of Lorentz transformations in the $x^3$ direction. The state $|\mathrm{i}\rangle$ has components $(p_{\mathrm{i}}^0, \mathbf{0}, p_{\mathrm{i}}^3)$ or $(\eta_{\mathrm{i}}, \mathbf{0}, h_{\mathrm{i}})$ and $\sinh\omega = p_{\mathrm{i}}^3/m$, $\mathrm{e}^{+\omega} = \sqrt{2}\eta_{\mathrm{i}}/m$.

so

$$S_{\mathrm{fi}} = \left\langle \mathrm{f}_0 \middle| \exp(\mathrm{i}\omega K_3) T \exp\left(-\mathrm{i}\int L(x)\,\mathrm{d}^4x\right)\exp(-\mathrm{i}\omega K_3) \middle| \mathrm{i}_0 \right\rangle . \tag{3.4}$$

Now

$$\exp(\mathrm{i}\omega K_3)\int L(x)\,\mathrm{d}^4x\exp(-\mathrm{i}\omega K_3)$$

$$= g\int \phi^\dagger(\epsilon\tau, z/\epsilon, \boldsymbol{x})\phi(\epsilon\tau, z/\epsilon, \boldsymbol{x})\phi^{\mathrm{ex}}(\tau, z, \boldsymbol{x})\,\mathrm{d}\tau\,\mathrm{d}z\,\mathrm{d}^2x$$

$$= \epsilon g\int \phi^\dagger(\epsilon\tau, z, x)\phi(\epsilon\tau, z, x)\phi^{\mathrm{ex}}(\tau, \epsilon z, x)\,\mathrm{d}\tau\,\mathrm{d}z\,\mathrm{d}^2x \tag{3.5}$$

where

$$\epsilon = \mathrm{e}^{-\omega}. \tag{3.6}$$

In the high energy limit, $\omega \to \infty$, the operators are evaluated at $\tau = 0$ and the external field at $z = 0$ (ie along the approximate path of the particle). Since

$$x^2 = 2\tau z - \boldsymbol{x}^2, \tag{3.7}$$

the operators are all mutually space-like, hence commute with one another, and the $T$ product can be ignored. Transforming back to the original frame, we obtain

$$S_{\mathrm{fi}} = \left\langle 0 \middle| a(\boldsymbol{p}_{\mathrm{f}}, \eta_{\mathrm{f}})\exp\left(-\mathrm{i}\int \chi(\boldsymbol{x})\rho(\boldsymbol{x})\,\mathrm{d}^2x\right) a^\dagger(\boldsymbol{p}_{\mathrm{i}}, \eta_{\mathrm{i}}) \middle| 0 \right\rangle, \tag{3.8}$$

where

$$\chi(\boldsymbol{x}) = g\int \phi^{\mathrm{ex}}(\tau, 0, \boldsymbol{x})\,\mathrm{d}\tau, \tag{3.9}$$

and

$$\rho(\boldsymbol{x}) = \int \phi^\dagger(0, z, \boldsymbol{x})\phi(0, z, \boldsymbol{x})\,\mathrm{d}z. \tag{3.10}$$

Using (2.11) and (2.9) this can be evaluated to give the relativistic eikonal form

$$S_{\mathrm{fi}} = (2\pi)2\eta_{\mathrm{i}}\delta(\eta_{\mathrm{i}} - \eta_{\mathrm{f}})\int \exp\left(\frac{-\mathrm{i}\chi(x)}{2\eta_{\mathrm{i}}}\right)\exp(\mathrm{i}\boldsymbol{\Delta}\,.\,\boldsymbol{x})\,\mathrm{d}^2x \tag{3.11}$$

where

$$\boldsymbol{\Delta} = \boldsymbol{p}_{\mathrm{i}} - \boldsymbol{p}_{\mathrm{f}}. \tag{3.12}$$

This is the result obtained by Cheng and Wu (1969) and by Bjorken *et al* (1971).

It is worth examining in a little more detail what the limit $\omega \to \infty$ implies in (3.5) and in the $S$ matrix element. The interaction Lagrangian

$$\mathscr{L} = \int \mathrm{d}^4x L(x),$$

which appears in the evaluation of the $S$ matrix elements when expanded into annihilation and creation operators contains terms of the form:

$$\frac{g}{(2\pi)^6}\int \mathrm{d}^2p'\,\mathrm{d}^2p\,\mathrm{d}^2q\,\mathrm{d}k\frac{\mathrm{d}p^{3'}}{2p^{0'}}\frac{\mathrm{d}p^3}{2p^0}$$

$$\times[\delta^2(\boldsymbol{p}'-\boldsymbol{p}-\boldsymbol{q})\delta(p^{3'}-p^3-k)\exp\{\mathrm{i}(p^{0'}-p^0-\nu)x^0\}a^\dagger(p')a(p)\phi_{\mathrm{ex}}(q,k,\nu)$$
$$+\delta^2(\boldsymbol{p}'+\boldsymbol{p}-\boldsymbol{q})\delta(p^{3'}+p^3-k)\exp\{\mathrm{i}(p^{0'}+p^0-\nu)x^0\}a^\dagger(p')b^\dagger(p)$$
$$\times\phi_{\mathrm{ex}}(\boldsymbol{q},k,\nu)+\ldots] \tag{3.13}$$

where

$$\phi_{\mathrm{ex}}(x)=\frac{1}{(2\pi)^3}\int \mathrm{d}^2q\,\mathrm{d}k\,\mathrm{d}\nu\phi_{\mathrm{ex}}(\boldsymbol{q},k,\nu)\exp\{-\mathrm{i}(x^0\nu-x^3k-\boldsymbol{q}\cdot\boldsymbol{x})\}. \tag{3.14}$$

We make the assumption that the external field contains frequencies and wavenumbers small compared to the energy and longitudinal momentum of the initial and final '$a$' particle in the rest frame of the external field. We can then separate the $p^3$, $p^{3'}$ range of integration into large positive (1+) and the rest (r)—small and/or negative. On account of the $\delta$ functions in (3.13) and our assumption about the significant range of $k$, if $p^3$ is the 1+ range $p^{3'}$ must also be in this range (scattering terms) or be large and negative (pair terms). The latter terms are damped out by the strongly oscillating exponential factor. Accordingly $L$ can be split into two commuting parts and we can write

$$S_{\mathrm{fi}}=\left\langle \mathrm{f}\left|T\exp\left(-\mathrm{i}\int L(x)\,\mathrm{d}^4x\right)_{(1+)}T\exp\left(-\mathrm{i}\int L(x)\,\mathrm{d}^4x\right)_{(r)}\right|\mathrm{i}\right\rangle,$$

where $T\exp(-\mathrm{i}\int L(x)\,\mathrm{d}^4x)_{(1+)}$ will contain only $a^\dagger a$ or $b^\dagger b$ (scattering) terms with $p^3$ large and positive, whereas $T\exp(-\mathrm{i}\int L(x)\,\mathrm{d}^4x)_{(r)}$ will also contain $a^\dagger b^\dagger$ and $ab$ (pair creating and annihilating) terms. Since the states $|\mathrm{i}\rangle$ and $|\mathrm{f}\rangle$ are states with $p_{\mathrm{i}}^3$ and $p_{\mathrm{f}}^3$ large and positive we can write

$$S_{\mathrm{fi}}=\left\langle 0\left|T\exp\left(-\mathrm{i}\int L(x)\,\mathrm{d}^4x\right)\right|0\right\rangle\left\langle \mathrm{f}\left|T\exp\left(-\mathrm{i}\int L(x)\,\mathrm{d}^4x\right)_{(1+)}\right|\mathrm{i}\right\rangle. \tag{3.15}$$

As we take $p_{\mathrm{i}}^3\to\infty$ the energies in the intermediate states will approach each other, that is, $p^{0'}-p^0\to 0$ and so the time dependence in the operator part of $T\exp(-\mathrm{i}\int L(x)\,\mathrm{d}^4x)_{(1+)}$ will disappear. The $T$ operation can then be ignored. Thus we again arrive at (3.11) apart from the factor

$$\langle 0|S|0\rangle\equiv\left\langle 0\left|T\exp\left(-\mathrm{i}\int L(x)\,\mathrm{d}^4x\right)\right|0\right\rangle. \tag{3.16}$$

For external fields which can create pairs this is not a simple phase, but an energy-independent factor multiplying all scattering and creation amplitudes to make allowance for the possibility of real pair creation by the external field from the vacuum (see Matthews and Salam 1953). This factor is essential for the unitarity of the $S$ matrix and is dependent only on the properties of the external field.

## 4. Scattering of two distinguishable particles

Let us consider the scattering (at high energies) of two scalar particles (a and b) due to the exchange of another scalar particle. We assume the interaction lagrangian density is now

$$L(x) = g_0\phi_a^\dagger(x)\phi_a(x)\phi_0(x) + g_b\phi_b^\dagger(x)\phi_b(x)\phi_0(x), \tag{4.1}$$

where $\phi_0(x)$ is a neutral scalar field of mass $\mu$. The particles a and b are assumed to have the same mass $m$ but are not antiparticles of each other. If we consider only those diagrams which arise due to the exchange of any number of $\phi_0$ type particles (Levy and Sucher 1969), the $S$ matrix element for a–b scattering can be written

$$S_{fi} = \left\langle p_f p_f' \middle| T \exp\left( -i \int L_{eff}(x, x')\, d^4x\, d^4x' \right) \middle| p_i p_i' \right\rangle, \tag{4.2}$$

where

$$L_{eff}(x, x') = \frac{g_a g_b}{2} j_a(x)\Delta_F(x-x') j_b(x') \tag{4.3}$$

where

$$j_i(x) = \phi_i^\dagger(x)\phi_i(x), \qquad i = a, b \tag{4.4}$$

and $\Delta_F(x-x')$ is the Feynman propagator for mass $\mu$,

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int \frac{\exp(-ikx)\, d^4k}{k^2-\mu^2+i\epsilon}. \tag{4.5}$$

We shall work in the CM system where the unprimed variables label the a particle (assumed right moving) and the primed variables labels the b particle (left moving). We then have (see footnote in § 3)

$$p_i = (p_i^0, \mathbf{0}, p_i^3) = (\eta_i, \mathbf{0}, h_i) \tag{4.6}$$

and

$$p_i' = (p_i^0, \mathbf{0}, -p_i^3) = (h_i, \mathbf{0}, \eta_i). \tag{4.7}$$

If we introduce the boosts $K_3^{(a)}$ and $K_3^{(b)}$ associated with the bare a and b particles respectively, we can write

$$|p_i, p_i'\rangle = \exp(-i\omega K_3^{(a)}) \exp(i\omega K_3^{(b)})|i_0\rangle, \tag{4.8}$$

where $|i_0\rangle$ is a state of an a and b particle both at rest. We do the same for the final state. We can now proceed as in the previous section, with the only difference that the $\tau'$ and $z'$ dependence of the $b$ field will be interchanged. Thus

$$\begin{aligned}\exp(-i\omega K_3^{(b)}) \exp(i\omega K_3^{(a)}) \int d^4x\, d^4x' L(x, x') \exp(-i\omega K_3^{(a)}) \exp(i\omega K_3^{(b)})\\ = \epsilon^2 \frac{g_a g_b}{2}\int d^4x\, d^4x' j_a(\epsilon\tau, z, \boldsymbol{x})\Delta_F(\tau-\epsilon\tau', \epsilon z - z', \boldsymbol{x}-\boldsymbol{x}') j_b(\tau', \epsilon z', \boldsymbol{x}').\end{aligned} \tag{4.9}$$

In the high energy limit ($\epsilon \to 0$) the $\tau$ argument of the $a$ operators and the $z'$ argument of the $b$ operators both tend to zero. Because of (3.7), in this limit all $a$ operators are mutually space-like, hence commute and similarly all $b$ operators are mutually space-like and commute. Of course all $a$ operators commute with all $b$ operators, so the $T$ operator in (4.2) can be ignored. Transforming back to the original frame we obtain

$$S_{fi} = \left\langle 0 \middle| a(\boldsymbol{p}_f, \eta_f) b(\boldsymbol{p}_f', h_f') \exp\left( -i \int d^2x\, d^2x' \rho(\boldsymbol{x}, \boldsymbol{x}')\chi(\boldsymbol{x}-\boldsymbol{x}') a^\dagger(\boldsymbol{p}_i, \eta_i) b^\dagger(\boldsymbol{p}_i', h_i') \right) \middle| 0 \right\rangle, \tag{4.10}$$

where

$$\rho(\boldsymbol{x}, \boldsymbol{x}') = \int \mathrm{d}z\, \mathrm{d}\tau' j_{\mathrm{a}}(0, z, \boldsymbol{x}) j_{\mathrm{b}}(\tau', 0, \boldsymbol{x}') \tag{4.11}$$

and

$$\chi(\boldsymbol{x}-\boldsymbol{x}') = \frac{g_{\mathrm{a}}g_{\mathrm{b}}}{2}\int \mathrm{d}\tau\, \mathrm{d}z' \Delta_{\mathrm{F}}(\tau, -z', \boldsymbol{x}-\boldsymbol{x}') = -\frac{g_{\mathrm{a}}g_{\mathrm{b}}}{4\pi} K_0(\mu|\boldsymbol{x}-\boldsymbol{x}'|). \tag{4.12}$$

If we now substitute the fields into (4.11) and employ the commutation relations—using the right-moving form for a particles, (2.9) and (2.11), and the left-moving form for the b particles, (2.14) and (2.15)—we find

$$S_{\mathrm{fi}} = (2\pi)^4 2s\, \delta(\eta_{\mathrm{i}}-\eta_{\mathrm{f}})\delta(h_{\mathrm{i}}'-h_{\mathrm{f}}')\, \delta^2(p_{\mathrm{i}}+p_{\mathrm{i}}'-p_{\mathrm{f}}-p_{\mathrm{f}}') \int \exp\left(-\mathrm{i}\frac{\chi(\boldsymbol{x})}{2s}\right) \exp(\mathrm{i}\boldsymbol{\Delta} \cdot \boldsymbol{x})\, \mathrm{d}^2 x \tag{4.13}$$

where

$$s = (p+p')^2 \simeq 2pp' = 2\eta_{\mathrm{i}}h_{\mathrm{i}}'. \tag{4.14}$$

This can be compared with the analogous result of Cheng and Wu (1969) for the case of electrodynamics. Note that the eikonal phase has a factor of $s^{-1}$. This is a consequence of the exchange particle having spin zero.

We can easily extend this result to the case of particles of higher spin being exchanged. We modify the interaction Lagrangian to be

$$L(x) = g_{\mathrm{a}}\phi_{\mathrm{a}}^{\dagger}(x)\overleftrightarrow{\partial}_{\mu}\phi_{\mathrm{a}}(x)A^{\mu}(x) + g_{\mathrm{b}}\phi_{\mathrm{b}}^{\dagger}(x)\overleftrightarrow{\partial}_{\mu}\phi_{\mathrm{b}}(x)A^{\mu}(x) \tag{4.15}$$

where

$$f\overleftrightarrow{\partial}_{\mu}g = f\frac{\partial}{\partial x^{\mu}}g - \frac{\partial}{\partial x^{\mu}}fg \tag{4.16}$$

and $A^{\mu}(x)$ is a neutral vector field of mass $\mu$. Again the $S$ matrix for a–b scattering which arises from the exchange of any number of vector mesons can be written as (4.2) where now

$$L_{\mathrm{eff}}(x, x') = \frac{g_{\mathrm{a}}g_{\mathrm{b}}}{2} j_{\mathrm{a}}^{\mu}(x)\Delta_{\mathrm{F}\mu\nu}(x-x') j_{\mathrm{b}}^{\nu}(x'), \tag{4.17}$$

where

$$j_i^{\mu}(x) = \phi_i^{\dagger}(x)\partial^{\mu}(x)\phi_i(x), \qquad i = \mathrm{a, b} \tag{4.18}$$

$$\Delta_{\mathrm{F}\mu\nu}^{(x)} = \left(g_{\mu\nu} - \frac{\partial_{\mu}\partial_{\nu}}{\mu^2}\right)\Delta_{\mathrm{F}}(x). \tag{4.19}$$

In the high energy limit only the $g_{\mu\nu}$ term will contribute. Everything proceeds as with scalar exchange except that the $\overleftrightarrow{\partial}_{\mu}$ in the currents will bring down extra factors of

$$4(\eta_{\mathrm{i}}h_{\mathrm{i}}' + h_{\mathrm{i}}\eta_{\mathrm{i}}' - f_{\mathrm{i}}f_{\mathrm{i}}') \simeq 4\eta_{\mathrm{i}}h_{\mathrm{i}}' = 2s. \tag{4.20}$$

In which case the $S$ matrix will be

$$S_{\mathrm{fi}} = (2\pi)^4 2s\delta(\eta_{\mathrm{i}}-\eta_{\mathrm{f}})\delta(h_{\mathrm{i}}'-h_{\mathrm{f}}')\delta^2(\boldsymbol{p}_{\mathrm{i}}+\boldsymbol{p}_{\mathrm{i}}'-\boldsymbol{p}_{\mathrm{f}}-\boldsymbol{p}_{\mathrm{f}}') \int \exp(-\mathrm{i}\chi(\boldsymbol{x})) \exp(\mathrm{i}\boldsymbol{\Delta} \cdot \boldsymbol{x})\, \mathrm{d}^2 x, \tag{4.21}$$

where $\chi(\boldsymbol{x})$ is still given by (4.12). Note that the dimensions of the coupling constants

$g_a g_b$ are different in the two cases. For the scalar case they have dimension of energy whereas they are dimensionless for the vector coupling.

We see how one can easily modify the result if the particle exchanged has spin $j$. Thus, the scattering of two distinguishable particles (a, b) via the multiple exchange of a particle of spin $j$ in the high energy limit is given by

$$S_{\mathrm{fi}} = (2\pi)^4 2s\delta(\eta_{\mathrm{i}} - \eta_{\mathrm{f}})\delta(h'_{\mathrm{i}} - h'_{\mathrm{f}})\delta^2(p_{\mathrm{i}} + p'_{\mathrm{i}} - p_{\mathrm{f}} - p'_{\mathrm{f}}) \int \exp(-\mathrm{i}s^{j-1}\chi(\boldsymbol{x})) \exp(\mathrm{i}\boldsymbol{\Delta} \cdot \boldsymbol{x})\, \mathrm{d}^2 x, \tag{4.22}$$

with $\chi(\boldsymbol{x})$ given by (4.12).

In all these cases the invariant $T$ matrix is given by

$$T_{\mathrm{fi}} = 2\mathrm{i}s \int \{\exp(-\mathrm{i}s^{j-1}\chi(\boldsymbol{x})) - 1\} \exp(\mathrm{i}\boldsymbol{\Delta} \cdot \boldsymbol{x})\, \mathrm{d}^2 x. \tag{4.23}$$

## 5. Exchange of Regge poles

From (4.22) we see that the $S$ matrix which arises from the multiple exchange of particles of spin $j$ has (at high energies) a simple dependence on spin. If instead of exchanging a particle we assumed exchange of a more complicated system of spin $j$ we need only replace the $\Delta_{\mathrm{F}}(x - x')$ by a more complicated function of $(x - x')$, say $V(x - x')$. Thus the final result is just what one would obtain for a and b particles interacting via an energy-dependent 'potential'

$$s^j V(x - x'), \tag{5.1}$$

and the eikonal phase is given (see (4.22)) by

$$s^{j-1} \int \mathrm{d}\tau\, \mathrm{d}z\, V(\tau, -z, \boldsymbol{x}). \tag{5.2}$$

The answer in this simple approximation is thus identical with that obtained from the exponentiation procedure defined by Cheng and Wu (1971). For this to be applied one need only know the Born approximation for the scattering, and by a simple Fourier transform one obtains the effective potential which enters into the eikonal phase.

As an example let us calculate the $T$ matrix for the multiple exchange of an absorptive (ie purely imaginary) Regge pole with arbitrary ordering of the end points (for a more elaborate discussion see Cardy 1971). We assume the 'Born approximation' to be

$$T_{\mathrm{B}} = -\mathrm{i}\beta s^{\alpha(t)}, \tag{5.3}$$

where for simplicity we take $\beta$ independent of $t$. Let us also assume a linear trajectory

$$\alpha(t) = \alpha_0 + \alpha' t, \tag{5.4}$$

with

$$\alpha' > 0. \tag{5.5}$$

The effective propagator $V(x)$ which would appear in (4.3) is now

$$V(x) = \frac{-\mathrm{i}\beta s^{\alpha_0 - 1}}{(2\pi)^4} \int \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \exp\{(\alpha' \ln s)\boldsymbol{p}^2\}\, \mathrm{d}^4 p, \tag{5.6}$$

where

$$p^2 = t = 2\eta h - \boldsymbol{p}^2. \tag{5.7}$$

The eikonal phase as defined in (4.12) is

$$\chi(\boldsymbol{x}) = \tfrac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}z\, V(\tau, -z, \boldsymbol{x}). \tag{5.8}$$

Note that the 'coupling constant' $\beta$ and, in contrast to (5.2), the $s$ dependence arising from the spin of the exchanged system is now included in $V$. Substituting (5.6), we obtain

$$\chi(\boldsymbol{x}) = \frac{-\mathrm{i}\beta s^{\alpha_0 - 1}}{2(2\pi)^2}\int \exp(\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{x})\exp\{-(\alpha' \ln s)\boldsymbol{p}^2\}\, \mathrm{d}^2 p \tag{5.9}$$

$$= -\frac{\mathrm{i}\beta}{\alpha'}\frac{s^{\alpha_0-1}}{(4\pi)^2}(\ln s)^{-1}\exp\left(\frac{-\boldsymbol{x}^2}{\alpha' \ln s}\right). \tag{5.10}$$

This phase is equivalent to that due to an absorptive gaussian potential whose strength and range both are energy dependent.

Thus the $T$ matrix is given (see (4.23)) by

$$T_{\mathrm{fi}} = 2\mathrm{i}s\int\left[\exp\left\{-g(s)\exp\left(\frac{-\boldsymbol{x}^2}{R^2(s)}\right)\right\} - 1\right]\exp(\mathrm{i}\boldsymbol{\Delta}\,.\,\boldsymbol{x})\,\mathrm{d}^2x, \tag{5.11}$$

where

$$g(s) = \frac{\beta}{\alpha'}\frac{s^{\alpha_0-1}}{(4\pi)^2}(\ln s)^{-1}, \tag{5.12}$$

$$R^2(s) = \alpha' \ln s. \tag{5.13}$$

For the case $\alpha_0 > 1$† and in the limit of large $s$ this may be approximated, as suggested by Froissart (1961), as

$$T_{\mathrm{fi}} = -2\mathrm{i}s\int_0^{\mathscr{R}} \exp(\mathrm{i}\boldsymbol{\Delta}\,.\,\boldsymbol{x})\,\mathrm{d}^2x, \tag{5.14}$$

where $\mathscr{R}$, is the radius of the equivalent black disc, defined by

$$g(s)\exp\left(\frac{-\mathscr{R}^2}{R^2}\right) = 1, \tag{5.15}$$

or

$$\mathscr{R}^2 = (\alpha_0 - 1)\alpha'(\ln s)^2. \tag{5.16}$$

For large $s$ we find

$$T_{\mathrm{fi}} = -2\mathrm{i}s\pi\frac{\mathscr{R}}{\sqrt{-t}}\mathrm{J}_1(\mathscr{R}\sqrt{-t}), \tag{5.17}$$

where

$$t = -(\boldsymbol{p}_{\mathrm{i}} - \boldsymbol{p}_{\mathrm{f}})^2. \tag{5.18}$$

† For the cases $\alpha_0 = 1$ or $< 1$ see Chang and Fishbane (1970), Chang and Yan (1970) and Hasslacher *et al* (1970).

The total cross section

$$\sigma_{\mathrm{T}} = \frac{-\operatorname{Im} T_{\mathrm{fi}}(t=0)}{s} = \pi\mathscr{R}^2, \tag{5.19}$$

$$= \pi(\alpha_0 - 1)\alpha'(\ln s)^2. \tag{5.20}$$

From (5.17) we see that the slope of the diffraction peak behaves like $(\alpha' \ln s)^2$. It is interesting to note that in this approximation the elastic amplitude and total cross section depend only on the effective radius $\mathscr{R}$. If instead of an energy-dependent range gaussian potential we used an exponential potential with constant range of the form found by Cheng and Wu (1970)

$$T_{\mathrm{B}} \simeq -\mathrm{i}s^{\alpha_0 - 1}\,\mathrm{e}^{-K(x)} \tag{5.21}$$

the $s$ dependence of (5.17) and (5.20) would have been the same (Zachariazen 1971).

## 6. Discussion

We have shown that the process of high energy elastic scattering due to the multiple exchange of particles of a given spin (or Regge poles) can be simply evaluated by modifying and extending a method introduced by Bjorken *et al* (1971) using light cone variables and applied by them to scatter in an external field. The main features of the approximation were examined in § 3. It is necessary to take a high energy limit in order that no particle–antiparticle pairs contribute and that the operators appearing in the $T$ product can all be evaluated either at a fixed $\tau$ or $z$. As we saw by examining the structure of equation (3.12) this is true if no longitudinal momentum is emitted or absorbed by the exchanged particles. Specifically, in equation (3.12) it was necessary that the phase factor $\exp\{\mathrm{i}(p_0' - p_0 - \nu)t\}$ be independent of $p_0$ and $p_0'$ which is true if $p_0 = p_0'$. This in turn implies in the high energy limit that $p^{3'} = p^3$. It then follows from the momentum conservation that no longitudinal momentum is transferred to the exchanged particle.

Applying the result to a simple absorptive Regge pole we reproduce earlier results (see Zachariazen 1971) on the high energy behaviour of elastic scattering and cross sections which reach the Froissart (1961) bound.

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